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Can Quantum Mechanics Be Reconciled With Cellular Automata?

Gerard 't Hooft[1,2]

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After a brief account of the GHZ version of the Bell inequalities, we indicate how fermionic fields can emerge in a description of statistical features in cellular automata. In square lattices, rotations over arbitrary angles can be formulated in terms of such fields, but it will be difficult to produce models with exact rotational invariance. Symmetries such as rotational symmetry will have to be central in attempts to produce realistic models.

KEY WORDS: quantum mechanics; cellular automata; rotational and translational symmetry; spin and the Bell inequalities.

1. A THOUGHT EXPERIMENT

Several speakers in this meeting express their optimism concerning the possibility to describe realistic models of the Universe in terms of deterministic "digital" scenarios. Most physicists, however, are acutely aware of quite severe obstacles against such views. It is important to contemplate these obstacles, even if one believes that they will eventually be removed. In general, they show that our world is such a strange place that "logical" analysis of our experiences appears to be impossible. I do believe that these are only appearances, but these facts invalidate many simple-minded ideas.

The common denominator is the "Bell inequality." Bell (1964) discovered that the outcomes of statistical experiments can violate inequalities that one can derive by assuming that every measurement can, in principle, be applied to any system of particles, even if only a small subset of experiments can be performed at the same time on the same system. His inequalities applied to the statistical outcome of such experiments. The version of the "Bell contradiction" that I like most is a more recent discovery (Greenberger *et al.*, 1989), where a setup is described that only produces certainties, not statistics, and these can only occur in quantum

[1] Institute for Theoretical Physics, Utrecht University, Utrecht, The Netherlands.
[2] To whom correspondence should be addressed at Spinoza Institute, Post Box 80.195, 3508 TD Utrecht, The Netherlands; e-mail: g.thooft@phys.uu.nl.

mechanical systems; in classical systems they are forbidden, and indeed, producing cellular automata with classical computers that mimic such strange effects, will always be difficult.

Our experience in the physical world is that setups can be made where particles can emerge in almost any desired wave function. Classically, one can think of a device that contains two dice, a red one and a green one. The machine is constructed in such a way that if one die emerges, say the red one, showing some number x (an integer between 1 and 6), then the other, the green one, will always show $y = 7 - x$. The two dice are shipped to two distant observers, without changing their orientations. If one observer sees, say, $x = 4$, he will know for certain that the other observer has $y = 3$.

In Quantum Mechanics, one can make more crazy devices of such kind (Greenberger *et al.*, 1989). A machine can be built that emits three particles, 1, 2, and 3, with spin $\frac{1}{2}$, say neutrons. The spin in the z-direction of each of these particles can have two values, $\pm\frac{1}{2}$. We omit the immaterial factor $\frac{1}{2}$, and say that there are three operators, called $\sigma_z^{(1)}$, $\sigma_z^{(2)}$, and $\sigma_z^{(3)}$. In a Hilbert space with altogether eight basic states, each of these operators has four (degenerate) eigenvalues $+1$ and four eigenvalues -1. We now assume that our device emits them either with all spins up ($\sigma_z^{(i)} = +1$), or all values down ($\sigma_z^{(i)} = -1$). More precisely, we assume that the "wave function" is

$$\psi = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle). \tag{1}$$

Now let's assume that the particles fly away towards three distant observers, living on different planets, and each of these observers will decide, on the spot, whether to measure either σ_x (the spin in the x-direction) or σ_y (the spin in the y-direction) of the particle that reaches him. The observers will not know in advance which measurement will be made by the other observers. In matrix form, the operators are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$

As is well-known, the observers are unable to measure both σ_x and σ_y.

Suppose that all observers had decided to measure σ_x. Then, with the wave function (1), it is easy to compute the expectation values

$$\left\langle \sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} \right\rangle = -1. \tag{3}$$

In other words, the measurements are completely correlated: if two observers measure $+1$, the third will surely find -1.

Similarly, there are correlations if one observer had measured σ_x while the two others measured σ_y:

$$\left\langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \right\rangle = +1, \tag{4}$$

$$\left\langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \right\rangle = +1, \tag{5}$$

$$\left\langle \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} \right\rangle = +1. \tag{6}$$

If only *one* of the observers, or all three of them, measured σ_y, one finds no correlations:

$$\left\langle \sigma_x^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} \right\rangle = \left\langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \right\rangle = 0. \tag{7}$$

One now could ask: what is the "ontological state" of the particles? Suppose we had determined empirically the correlations (4), (5), and (6). If we knew for certain that the particles will always behave this way, we could say: well then, multiply the three expressions together. Since all measurements give either $+1$ or -1, and for each particle σ_x is measured only once, while σ_y is measured twice, one would expect that the product of the σ_x measurements should always be $+1$, completely in conflict with Eq. (3).

One must conclude from this experiment, of which several versions have really been carried out, that it is impossible to have a particle and say: if I would measure σ_x the outcome would be this, and if I would measure σ_y, the outcome would be that. Our problem with cellular automaton models is that one would very much be inclined to allow for such attributions to a particle. According to Quantum Mechanics, this is not allowed.

2. TRANSLATIONS

One of the key assumptions in the above scenario is that replacing a measurement device by one that is rotated 90° is allowed without affecting in any way the "ontological" state of the particle that is being measured. In gravity theories, this might be questioned: rotating any macroscopic device may cause the emission of ripples of gravitational waves, enough to disturb the particle in question. Rotation is one of the simplest examples of a symmetry transformation. The experiment above assumed that I can rotate a device *locally*, without simultaneously rotating the particle that is on its way to the apparatus. "Spin" indeed refers to how an object responds upon a rotation. It cannot be an ontologically impeccable property of a particle. How can rotations, in particular rotations over arbitrary angles, be viewed in a cellular automaton, which after all usually requires the introduction of a lattice? Lattices usually do not allow for more rotational symmetry than rotation over fixed angles, typically 90°.

Before discussing rotation, I first consider translations. If you have a discrete lattice, at first sight only translations over some integral multiple of the unit lattice link size a are allowed. But the knowledge of a little Quantum Field Theory allows us to do better.

Suppose, for simplicity, that we have a sequence of ones and zeros on a one-dimensional lattice. The translation operator $T(x)$ is defined to effectuate a displacement of all zeros and ones by a distance x, if $x = Na$, and N is integer. How do we define $T(x)$ if x/a is not integer? In particle theory, we can do this: first, the operator $\psi(x)$, where x is a lattice site, is defined as follows:

$$\begin{aligned}\psi(x)|1\rangle_x &= (-1)^{N(x)}|0\rangle_x,\\ \psi(x)|0\rangle_x &= 0,\\ \psi^\dagger(x)|1\rangle_x &= 0,\\ \psi^\dagger(x)|0\rangle_x &= (-1)^{N(x)}|1\rangle_x.\end{aligned} \tag{8}$$

Here, the suffix x indicates that the entry at the lattice site x is the one inside the brackets, 0 or 1, and only that entry is affected. The quantity $N(x)$ is defined to be the *total number of ones at the left of the site* x. The operator $\psi^\dagger$ is the Hermitean conjugate of ψ.

It is easy to convince oneself that the product $\psi^\dagger(x)\psi(x)$ is an operator that leaves the state unchanged, giving one if there is a one at the site x, and zero otherwise: $\psi^\dagger(x)\psi(x)|\sigma\rangle_x = \sigma|\sigma\rangle_x$. Now, notice

$$\psi^2(x) = 0, \tag{9a}$$

$$\psi(x)\psi(x') = -\psi(x')\psi(x), \tag{9b}$$

$$\psi(x)\psi^\dagger(x') + \psi^\dagger(x')\psi(x) = \delta(x, x'). \tag{9c}$$

Notice that the minus sign in (9b) and the plus sign in (9c) follow from the $(-1)^{N(x)}$ in (8). They ensure that (9a) is a special case of (9b). What is nice about these equations is that you can Fourier transform $\psi(x)$:

$$\psi(x) = (a/2\pi)^{1/2}\int_{-\pi/a}^{+\pi/a} dp\, e^{ipx}\hat\psi(p), \tag{10}$$

after which $\hat\psi(p)$ obeys equations very similar to (9a):

$$\hat\psi^2(p) = 0, \tag{11a}$$

$$\hat\psi(p)\hat\psi(p') = -\hat\psi(p')\hat\psi(p), \tag{11b}$$

$$\hat\psi(p)\hat\psi^\dagger(p') + \hat\psi^\dagger(p')\hat\psi(p) = \delta(p - p'). \tag{11c}$$

$\hat\psi(p)$ is said to be the operator that annihilates a “particle with momentum p.” These particles are fermions; you can’t have two of them at the same place, either in position space or in momentum space, because $\hat\psi^2(p) = 0$.

In Fourier space, a translation $T(b)$ simply multiples $\hat{\psi}(p)$ with a factor e^{ipb}. *But now it is obvious that the same definition of a translation can be given if b is not a multiple of the lattice length*! Fourier transforming back to position space then gives the new definition of $\psi(x)$ in terms of the old one:

$$T(b)\colon \psi'(x) = \sum_{x'} \frac{a \sin(\pi(x - x' - b)/a)}{\pi(x - x' - b)} \psi(x'). \tag{12}$$

In the limit where $b \to Na$, with N integer, this is just the usual displacement. In the other cases, we see that $\psi'(x)$ produces a *linear (quantum) combination of states*!

3. ROTATIONS

Defining a rotation $R(\phi)$ for any (fractional) angle ϕ can be done in similar ways, but is not quite that easy. Imagine that we define an operator $\psi(x, y)$ in a two-dimensional position space. The definition is just as in Eqs. (8), except that the function $N(x, y)$ is a bit more awkward to define:

$N(x)$ is the number of ones at all sites (x', y') such that either $y' < y$ or ($y' = y$, and $x' < x$).

Fourier transforming goes as usual, but now, Fourier space is the space of values (p_x, p_y) with $|p_x| < \pi/a$ and $|p_y| < \pi/a$, in other words: a square.

If we rotate this square by an angle ϕ that is not a multiple of 90° then the edges do not match (see Fig. 1). There are several things one can do now. The whole point is that, usually, we are interested only in large scale phenomena. These are the phenomena that usually correspond to very small values for p_x and p_y. Thus, if we make sure that all points inside the inscribed circle of this square are rotated simply by an angle ϕ, then the most relevant features all rotate as required. The prescriptions at the edges will be more artificial and model dependent, but have little effect on large-scale phenomena.

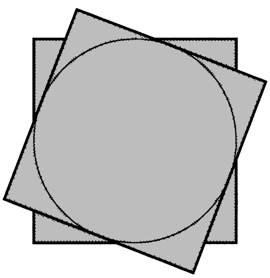

Fig. 1. A rotation in Fourier space.

It is important that successive applications of translations and rotations have the usual effects. This is called group theory. For instance,

$$T(\vec{b})R(\phi) = R(\phi)T(\Omega\vec{b}), \qquad (13)$$

where Ω is the rotation over an angle ϕ. Equation (13) cannot be obeyed exactly because the edges of the square cannot be made to match. One of our worries will therefore be that we will have to explain an apparently perfect rotational symmetry in the world that we are trying to describe.

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